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Regular connections among generalized connections

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Abstract

The properties of the space $\mathcal{A}$ of regular connections as a subset of the space $\overline{\mathcal{A}}$ of generalized connections in the Ashtekar framework are studied. For every choice of compact structure group and smoothness category for the paths, it is determined whether $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$ or not. Moreover, it is proven that $\mathcal{A}$ has Ashtekar–Lewandowski measure zero for every non-trivial structure group and every smoothness category. The analogous results hold for gauge orbits instead of connections.

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1. Introduction

One of the most important quantization methods is the functional integral approach. There, in a first step, one determines a physical Euclidean measure on the configuration space and reconstructs then, in a second step, the Hamiltonian theory using a kind of Osterwalder–Schrader procedure. In the case of pure gauge field theories, the configuration space is the space $\mathcal{A}/\mathcal{G}$ of smooth connections (i.e. gauge fields) modulo smooth gauge transformations in some principal fibre bundle P over the (space[-time]) manifold M with the structure group $\mathbf{G}$. However, in general, the structure of $\mathcal{A}/\mathcal{G}$ is very complicated: It is a non-affine, non-compact, not finite-dimensional space and not a manifold. Therefore, when defining measures there, enormous problems appeared that have been solved to date

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only partially. To avoid some of these problems, Ashtekar et al. [2,5,4,3] proposed to extend the configuration space by distributional gauge orbits. Using C^*-algebraic techniques, this space $\overline{\mathcal{A}/\mathcal{G}}$ could be interpreted as the compact spectrum of the C^*-algebra generated by the Wilson loops based on piecewise analytic paths. Rendall [17] showed that $\mathcal{A}/\mathcal{G}$ can be densely imbedded into $\overline{\mathcal{A}/\mathcal{G}}$. This coincides fully with the expectations made in other rigorous functional integral approaches, e.g. in the Wiener-integral study of the diffusion equation. Later on, also the spaces $\mathcal{A}$ and $\mathcal{G}$ of smooth connections and gauge transformations, respectively, have been enlarged by distributional objects, leading to the spaces $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$. Using projective-limit techniques it has been shown that $\overline{\mathcal{A}}/\overline{\mathcal{G}} \cong \overline{\mathcal{A}/\mathcal{G}}$.

However, there still had been a problem: in view of the desired applicability of the new approach to quantum gravity, due to its diffeomorphism invariance, one should consider at least smooth paths for the arguments of the parallel transports. The main problem from the technical side is the following: in contrast to the piecewise analytical category two paths now can have infinitely many intersection points without sharing just a complete interval. In other words, two finite graphs need no longer be contained in a bigger third graph being again finite. This problem has been first cured by Baez and Sawin [7,6] in the immersive smooth case using so-called webs. Recently [11,8], it has been shown that using so-called hyphs all smoothness categories can be handled at the same footing, whereas webs and graphs now are special kinds of hyphs.

The knowledge about the rôle of regular (i.e. smooth) connections in these non-analytic frameworks, however, is still quite limited. In the case of webs, only for the case of connected and semisimple structure groups **G** it is known that $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$ (and consequently $\mathcal{A}/\mathcal{G}$ in $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ as well). Therefore, we will now study in this article the properties of the spaces $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{A}/\mathcal{G}$ viewed as subspaces of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$, respectively, in more detail within the hyph framework. The outline of this main body of the paper will be as follows. First we will discuss how these embeddings may depend on the necessary choice of a specific (typically non-smooth) trivialization of P. It will turn out that all possible embeddings are in a certain sense equivalent. Afterwards, we will prove that $\mathcal{G}$ is dense in $\overline{\mathcal{G}}$ iff **G** is connected. The corresponding criterion for $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$, however, will be more difficult. As we will see, whether the denseness is given or not, does crucially depend on the structure group **G** and on the used smoothness category for the paths. The most important result will be that in the non-analytic framework the denseness is at most be given for semisimple **G**. That section is followed by a discussion how to modify the definitions of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ to get possibly the desired denseness. Finally, we will generalize the theorem of Marolf and Mourão about the Ashtekar–Lewandowski measure of $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$ to the case of general smoothness of paths.

We remark that an application of the denseness results of the present article to the C^*-algebraic formulation of the Ashtekar framework can be found in the paper [1] by Abbati and Manià.

2. Preliminaries

In the section, we briefly recall the basic definitions and conventions used in this paper. General expositions can be found in [5,4,3] for the analytic framework and in [9,11,8] for arbitrary smoothness classes. The notion and the properties of hyphs are discussed in [8,9].

Let now $\mathbf{G}$ be some arbitrary compact Lie group, M be a connected manifold having at least dimension 2 and m be some arbitrary, but fixed point in M. (The restriction to $\dim M \geq 2$ is only due to technical reasons.) $P(M, \mathbf{G})$—or, shortly, P—denotes some principal fibre bundle $\pi : P \to M$ with structure group $\mathbf{G}$, and P_x denotes the fibre $\pi^{-1}(x)$ over $x \in M$ in P. Next, we choose once and for all some smoothness type C^r for the paths with $r \in \mathbb{N}_+$, $r = \infty$ ("smooth") or $r = \omega$ ("analytic"), of course, with r not being larger than the smoothness category of M, and decide whether we will consider only immersive paths or also non-immersive paths. Now, $\mathcal{P}$ denotes the set of all (finite) paths in M. $\mathcal{P}$ is (after imposing the standard equivalence relation, i.e. saying that reparametrizations and insertions/deletions of retracings are irrelevant) a groupoid. A graph is a finite set of edges (i.e. possibly closed, but elsewhere non-selfintersecting paths) that intersect each other at most in their endpoints. The subgroupoid generated by the paths in a graph Γ will be denoted by $\mathcal{P}_\Gamma$. Graphs are ordered in the natural way: $\Gamma' \leq \Gamma''$ iff $\mathcal{P}_{\Gamma'} \subseteq \mathcal{P}_{\Gamma''}$. The set $\overline{\mathcal{A}}$ of generalized connections $\bar{A}$ is now defined by

$$\overline{\mathcal{A}} := \varprojlim_\Gamma \overline{\mathcal{A}}_\Gamma \cong \mathrm{Hom}(\mathcal{P}, \mathbf{G})$$

with $\overline{\mathcal{A}}_{\boldsymbol{\gamma}} := \mathrm{Hom}(\mathcal{P}_{\boldsymbol{\gamma}}, \mathbf{G}) \cong \mathbf{G}^{\#\boldsymbol{\gamma}}$ for all finite sets $\boldsymbol{\gamma}$ of paths. (Often $\bar{A}$ is written synonymously as $h_{\bar{A}}$ to stress on the homomorphy property.) Correspondingly, the set $\overline{\mathcal{G}}$ of all generalized gauge transformations $\bar{g}$ is defined by

$$\overline{\mathcal{G}} := \varprojlim_\Gamma \overline{\mathcal{G}}_\Gamma \cong \mathrm{Maps}(M, \mathbf{G})$$

with $\overline{\mathcal{G}}_{\boldsymbol{\gamma}} := \mathrm{Maps}(\mathbf{V}(\boldsymbol{\gamma}), \mathbf{G}) \cong \mathbf{G}^{\#\mathbf{V}(\boldsymbol{\gamma})}$ for all finite $\boldsymbol{\gamma} \subseteq \mathcal{P}$, where $\mathbf{V}(\boldsymbol{\gamma})$ denotes the set of all end points of the paths in $\boldsymbol{\gamma}$. The value of $\bar{g}$ in $x \in M$ is denoted by $\bar{g}_x \in \mathbf{G}$ or sometimes shortly by g_x. The space $\overline{\mathcal{G}}$ acts continuously on $\overline{\mathcal{A}}$ via

$$h_{\bar{A}\circ\bar{g}}(\gamma) = \bar{g}_{\gamma(0)}^{-1} h_{\bar{A}}(\gamma) \bar{g}_{\gamma(1)} \quad \text{for all paths } \gamma$$

yielding the factor space $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ of generalized gauge orbits. $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ are compact Hausdorff spaces. Moreover, $\overline{\mathcal{G}}$ acts on $\overline{\mathcal{G}}$ continuously by conjugation: $\bar{g} \circ \bar{g}' := \bar{g}'^{-1} \cdot \bar{g} \cdot \bar{g}'$. This action is compatible with the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$, i.e. $(\bar{A} \circ \bar{g}) \circ \bar{g}' = (\bar{A} \circ \bar{g}') \circ (\bar{g} \circ \bar{g}')$. (Note, that the action in both cases is always from the right.)

A hyph υ is a finite (ordered) set of edges $e_1, \ldots, e_{\#\upsilon}$, where every $e_i \in \upsilon$ possesses some "free" point. This means, for at least one direction none of the segments of e_i starting in that point in this direction is a full segment of some of the $e_1, \ldots, e_{i-1}$. The set of hyphs is ordered analogously to the set of graphs. In contrast to the case of graphs, this ordering is a direct ordering in the case of hyphs for *every* smoothness category, i.e. for each two hyphs there is always some third hyph containing both. Nevertheless, we have

$$\overline{\mathcal{A}} \cong \varprojlim_\upsilon \overline{\mathcal{A}}_\upsilon, \qquad \overline{\mathcal{G}} \cong \varprojlim_\upsilon \overline{\mathcal{G}}_\upsilon \quad \text{and} \quad \overline{\mathcal{A}}/\overline{\mathcal{G}} \cong \varprojlim_\upsilon \overline{\mathcal{A}}_\upsilon/\overline{\mathcal{G}}_\upsilon.$$

The corresponding continuous projections to the constituents of the projective limits are in all these three cases denoted by π_υ, respectively. It has been proven, that π_υ is surjective for all hyphs υ.

Finally, the Ashtekar–Lewandowski measure μ_0 is the unique regular Borel measure on $\overline{\mathcal{A}}$ whose push-forward $(\pi_\upsilon)_*\mu_0$ to $\overline{\mathcal{A}}_\upsilon$ coincides with the Haar measure there for every hyph υ. Since μ_0 is $\overline{\mathcal{G}}$-invariant, it can be seen as a measure on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ as well.

3. Embeddings

In the projective-limit approach to Ashtekar connections, one needs a certain global trivialization of the underlying principal fibre bundle. That this trivialization can be chosen smooth is, of course, only possible if we are given a globally trivial bundle. However, for Ashtekar connections we can take *any* trivialization, i.e. one for every fibre separately. It is, therefore, necessary to investigate how the choice of trivializations influences the embedding of the smooth objects into their extensions. (Some investigations have been made already in [3].) But, as we will see this influence can be neglected. More precisely, for any two trivializations we find some isomorphism of $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$, respectively, that maps the one embedding to the other and respects the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$. Consequently, we will see that the embedding of $\mathcal{A}/\mathcal{G}$ into $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ is even completely independent of the choice of the embedding—may the trivialization be non-smooth everywhere.

We start with the following definition.

Definition 3.1. Let $\Xi = \{\Xi_x : P_x \to \mathbf{G} | x \in M\}$ be a set of fibre trivializations and denote the parallel transports according to $A \in \mathcal{A}$ along the path $\gamma \in \mathcal{P}$ by $\tau_{\gamma,A} : P_{\gamma(0)} \to P_{\gamma(1)}$.

The *embedding* ι_Ξ *of the regular gauge theory* into the generalized gauge theory corresponding to Ξ consists of the following three mappings all again denoted by ι_Ξ:

1. $\iota_\Xi : \mathcal{A} \to \overline{\mathcal{A}}$ with $h_{\iota_\Xi(A)}(\gamma) := (\Xi_{\gamma(1)} \circ \tau_{\gamma,A} \circ \Xi_{\gamma(0)}^{-1})(e_\mathbf{G}) \in \mathbf{G}$.
2. $\iota_\Xi : \mathcal{G} \to \overline{\mathcal{G}}$ with $(\iota_\Xi(g))(x) := (\Xi_x \circ g \circ \Xi_x^{-1})(e_\mathbf{G}) \in \mathbf{G}$.
3. $\iota_\Xi : \mathcal{A}/\mathcal{G} \to \overline{\mathcal{A}}/\overline{\mathcal{G}}$ with $\iota_\Xi([A]_\mathcal{G}) := [\iota_\Xi(A)]_{\overline{\mathcal{G}}}$.

Recall that Ξ_x, in general, does not depend continuously on x.

Proposition 3.1. *For every set* Ξ *of fibre trivializations,* $\iota_\Xi : \mathcal{A} \to \overline{\mathcal{A}}$ *and* $\iota_\Xi : \mathcal{G} \to \overline{\mathcal{G}}$ *and* $\iota_\Xi : \mathcal{A}/\mathcal{G} \to \overline{\mathcal{A}}/\overline{\mathcal{G}}$ *are well-defined mappings. Moreover, the second one is a group homomorphism.*

Proof.

1. $\iota_\Xi : \mathcal{A} \to \overline{\mathcal{A}}$ is well defined, since for all composable $\gamma, \delta \in \mathcal{P}$ we have

$$\begin{aligned} &h_{\iota_\Xi(A)}(\gamma \circ \delta) \\ &\quad = (\Xi_{(\gamma\circ\delta)(1)} \circ \tau_{\gamma\circ\delta,A} \circ \Xi_{(\gamma\circ\delta)(0)}^{-1})(e_\mathbf{G}) \\ &\quad = (\Xi_{\delta(1)} \circ \tau_{\delta,A} \circ \tau_{\gamma,A} \circ \Xi_{\gamma(0)}^{-1})(e_\mathbf{G}) \\ &\quad = (\Xi_{\delta(1)} \circ \tau_{\delta,A} \circ \Xi_{\delta(0)}^{-1} \circ \Xi_{\gamma(1)} \circ \tau_{\gamma,A} \circ \Xi_{\gamma(0)}^{-1})(e_\mathbf{G}) \\ &\quad = (\Xi_{\gamma(1)} \circ \tau_{\gamma,A} \circ \Xi_{\gamma(0)}^{-1})(e_\mathbf{G}) \cdot (\Xi_{\delta(1)} \circ \tau_{\delta,A} \circ \Xi_{\delta(0)}^{-1})(e_\mathbf{G}) \\ &\quad = h_{\iota_\Xi(A)}(\gamma) h_{\iota_\Xi(A)}(\delta) \end{aligned}$$

   by the invariance of the parallel transport under the action of $\mathbf{G}$ on P.
2. $\iota_\Xi : \mathcal{G} \to \overline{\mathcal{G}}$ is obviously well defined, and moreover a group homomorphism.

3. $\iota_\Xi : \mathcal{A}/\mathcal{G} \to \overline{\mathcal{A}}/\overline{\mathcal{G}}$ is well defined, since for all paths $\gamma \in \mathcal{P}$

$$\begin{aligned}
&h_{\iota_\Xi(A\circ g)}(\gamma)\\
&= (\Xi_{\gamma(1)} \circ \tau_{\gamma,A\circ g} \circ \Xi^{-1}_{\gamma(0)})(e_{\mathbf{G}})\\
&= (\Xi_{\gamma(1)} \circ g \circ \tau_{\gamma,A} \circ g^{-1} \circ \Xi^{-1}_{\gamma(0)})(e_{\mathbf{G}})\\
&= (\Xi_{\gamma(1)} \circ g \circ \Xi^{-1}_{\gamma(1)} \circ \Xi_{\gamma(1)} \circ \tau_{\gamma,A} \circ \Xi^{-1}_{\gamma(0)} \circ \Xi_{\gamma(0)} \circ g^{-1} \circ \Xi^{-1}_{\gamma(0)})(e_{\mathbf{G}})\\
&= (\Xi_{\gamma(0)} \circ g^{-1} \circ \Xi^{-1}_{\gamma(0)})(e_{\mathbf{G}})\cdot(\Xi_{\gamma(1)} \circ \tau_{\gamma,A} \circ \Xi^{-1}_{\gamma(0)})(e_{\mathbf{G}})\cdot(\Xi_{\gamma(1)}\circ g \circ \Xi^{-1}_{\gamma(1)})(e_{\mathbf{G}})\\
&= (\iota_\Xi(g))(\gamma(0))^{-1} \cdot h_{\iota_\Xi(A)}(\gamma) \cdot (\iota_\Xi(g))(\gamma(1))\\
&= h_{\iota_\Xi(A)\circ\iota_\Xi(g)}(\gamma). \qquad \square
\end{aligned}$$

Definition 3.2. Let Ξ_1 and Ξ_2 be two trivializations.

Then ι_{Ξ_1} and ι_{Ξ_2} are called *equivalent* iff there is some $\bar{g}^{\Xi_1}_{\Xi_2} \in \overline{\mathcal{G}}$ such that

1. $\iota_{\Xi_2}(A) = \iota_{\Xi_1}(A) \circ \bar{g}^{\Xi_1}_{\Xi_2}$ for all $A \in \mathcal{A}$ and
2. $\iota_{\Xi_2}(g) = \iota_{\Xi_1}(g) \circ \bar{g}^{\Xi_1}_{\Xi_2}$ for all $g \in \mathcal{G}$.

Corollary 3.2. *If two trivializations are equivalent, then the corresponding embeddings of $\mathcal{A}/\mathcal{G}$ coincide.*

Proof. By definition, for two equivalent trivializations Ξ_1 and Ξ_2 there is some $\bar{g} \in \overline{\mathcal{G}}$ such that $\iota_{\Xi_2}(A) = \iota_{\Xi_1}(A) \circ \bar{g}$, hence $\iota_{\Xi_2}([A]_{\mathcal{G}}) \equiv [\iota_{\Xi_2}(A)]_{\overline{\mathcal{G}}} = [\iota_{\Xi_1}(A)]_{\overline{\mathcal{G}}} \equiv \iota_{\Xi_1}([A]_{\mathcal{G}})$ for all $A \in \mathcal{A}$. $\square$

As it was to be expected, we have the following theorem.

Theorem 3.3. *All embeddings are mutually equivalent.*

Proof. Let Ξ_1 and Ξ_2 be some trivializations. Define $g_x := [((\Xi_1)_x \circ (\Xi_2)^{-1}_x)(e_{\mathbf{G}})]^{-1} \in \mathbf{G}$ for $x \in M$ and set $\bar{g}^{\Xi_1}_{\Xi_2} := (g_x)_{x\in M}$. Then we have for all $A \in \mathcal{A}$ and all $\gamma \in \mathcal{P}$

$$\begin{aligned}
&h_{\iota_{\Xi_1}(A)\circ\bar{g}^{\Xi_1}_{\Xi_2}}(\gamma)\\
&= (\bar{g}^{\Xi_1}_{\Xi_2})^{-1}_{\gamma(0)} h_{\iota_{\Xi_1}(A)}(\gamma)(\bar{g}^{\Xi_1}_{\Xi_2})_{\gamma(1)}\\
&= [((\Xi_1)_{\gamma(0)} \circ (\Xi_2)^{-1}_{\gamma(0)})(e_{\mathbf{G}})]((\Xi_1)_{\gamma(1)} \circ \tau_{\gamma,A} \circ (\Xi_1)^{-1}_{\gamma(0)})(e_{\mathbf{G}})\\
&\quad\cdot[((\Xi_1)_{\gamma(1)} \circ (\Xi_2)^{-1}_{\gamma(1)})(e_{\mathbf{G}})]^{-1}\\
&= ((\Xi_2)_{\gamma(1)} \circ (\Xi_1)^{-1}_{\gamma(1)} \circ (\Xi_1)_{\gamma(1)} \circ \tau_{\gamma,A} \circ (\Xi_1)^{-1}_{\gamma(0)} \circ (\Xi_1)_{\gamma(0)} \circ (\Xi_2)^{-1}_{\gamma(0)})(e_{\mathbf{G}})\\
&= h_{\iota_{\Xi_2}(A)}(\gamma),
\end{aligned}$$

hence $\iota_{\Xi_2}(A) = \iota_{\Xi_1}(A) \circ \bar{g}^{\Xi_1}_{\Xi_2}$. It is easy to see that analogously $\iota_{\Xi_2}(e) = \iota_{\Xi_1}(e) \circ \bar{g}^{\Xi_1}_{\Xi_2}$. $\square$

Until now we have not justified the notion “embedding” for ι_Ξ. This will be caught up on next.

Proposition 3.4. *For every set* Ξ *of fibre trivializations,* $\iota_\Xi : \mathcal{A} \to \overline{\mathcal{A}}$ *and* $\iota_\Xi : \mathcal{G} \to \overline{\mathcal{G}}$ *and* $\iota_\Xi : \mathcal{A}/\mathcal{G} \to \overline{\mathcal{A}}/\overline{\mathcal{G}}$ *are injective.*

Proof.

1. *Injectivity of* $\iota_\Xi : \mathcal{A} \to \overline{\mathcal{A}}$. For this, let A_1 and A_2 be two connections with $h_{\iota_\Xi(A_1)}(\gamma) = h_{\iota_\Xi(A_2)}(\gamma)$ for all $\gamma \in \mathcal{P}$. Assume $A_1 \neq A_2$. Then there is some $p \in P$ and some tangent vector $X \in T_p P$ such that X is horizontal w.r.t. A_1, but not w.r.t. A_2. If, however, γ is some path through $\pi(p)$, whose tangent vector in $\pi(p)$ equals $\pi_* X$, then the horizontal lift $\tilde{\gamma}_1$ of γ w.r.t. A_1 has to be different from the horizontal lift $\tilde{\gamma}_2$ w.r.t. A_2. Hence (at least along a suitable subpath of γ) the corresponding parallel transports have to be different as well, in contradiction to $h_{\iota_\Xi(A_1)} = h_{\iota_\Xi(A_2)}$ or to the homeomorphy property of Ξ_x for some $x \in M$.
2. *Injectivity of* $\iota_\Xi : \mathcal{G} \to \overline{\mathcal{G}}$. This is obvious.
3. *Injectivity of* $\iota_\Xi : \mathcal{A}/\mathcal{G} \to \overline{\mathcal{A}}/\overline{\mathcal{G}}$. Let $[A_1]$ and $[A_2]$ be in $\mathcal{A}/\mathcal{G}$ with $\iota_\Xi([A_1]) = \iota_\Xi([A_2])$. This means, $\iota_\Xi(A_2) = \iota_\Xi(A_1) \circ \bar{g}_\Xi$ for some $\bar{g}_\Xi \in \overline{\mathcal{G}}$ depending on Ξ. By Theorem 3.3, we can choose $\bar{g}_\Xi$ such that $\bar{g}_{\Xi_2} = \bar{g}_{\Xi_1} \circ \bar{g}^{\Xi_1}_{\Xi_2}$ with $(\bar{g}^{\Xi_1}_{\Xi_2})_x := [((\Xi_1)_x \circ (\Xi_2)_x^{-1})(e_{\mathbf{G}})]^{-1}$ for every two trivializations Ξ_1 and Ξ_2. We have to show that $\bar{g}_\Xi$ can be chosen regular, i.e. $\bar{g}_\Xi \in \iota_\Xi(\mathcal{G})$.

   For this, let $\gamma(t)$ be some path in M with initial point x. The relation $h_{\iota_\Xi(A_2)}(\gamma) = (\bar{g}_\Xi)_x^{-1} h_{\iota_\Xi(A_1)}(\gamma)(\bar{g}_\Xi)_y$ for all $\gamma \in \mathcal{P}_{xy}$ guarantees that $(\bar{g}_\Xi)_{\gamma(t)} = h_{\iota_\Xi(A_1)}(\gamma|_{[0,t]})^{-1} (\bar{g}_\Xi)_x h_{\iota_\Xi(A_2)}(\gamma|_{[0,t]})$. Since we reach every point in M by some path γ starting in x and since Ξ is a bijection between $M \times \mathbf{G}$ and $\pi^{-1}(M) = P$, there is a unique (not necessarily differentiable) gauge transform $g_\Xi : P \to P$ with $(\iota_\Xi(g_\Xi))(y) \equiv (\Xi_y \circ g_\Xi \circ \Xi_y^{-1})(e_{\mathbf{G}}) = (\bar{g}_\Xi)_y$ for all $y \in M$.

   Now we assume that Ξ is C^r over some open $U \subseteq M$. Then $h_{\iota_\Xi(A)}(\gamma|_{[0,t]})$ depends differentiably on t for every $\gamma \in \mathcal{P}$ with $\operatorname{im} \gamma \subseteq U$ and, consequently, $(g_\Xi)_{\gamma(t)}$ is differentiable as well. Using that Ξ is a diffeomorphism between $U \times \mathbf{G}$ and $\pi^{-1}(U) \subseteq P$, we see that $g_\Xi|_{\pi^{-1}(U)}$ is even differentiable.

   Since around every point in M there is some differentiable trivialization, we can define a global gauge transform $g : P \to P$ by $g(p) := g_\Xi(p)$ for $p \in \pi^{-1}(U)$ if Ξ is any differentiable trivialization over a neighbourhood U of $x \in M$. Let now Ξ and Ξ' be two differentiable trivializations over U. For $\gamma \in \mathcal{P}$ contained in U and having base point x, we have

$$\begin{aligned}
\iota_\Xi(g_{\Xi'})(\gamma(t)) &= (\iota_{\Xi'}(g_{\Xi'}) \circ \bar{g}^{\Xi'}_{\Xi})(\gamma(t)) \\
&= (\bar{g}^{\Xi'}_{\Xi})^{-1}_{\gamma(t)} h_{\iota_{\Xi'}(A_1)}(\gamma|_{[0,t]})^{-1} (\bar{g}_{\Xi'})_x h_{\iota_{\Xi'}(A_2)}(\gamma|_{[0,t]}) (\bar{g}^{\Xi'}_{\Xi})_{\gamma(t)} \\
&= h_{\iota_\Xi(A_1)}(\gamma|_{[0,t]})^{-1} (\bar{g}^{\Xi'}_{\Xi})^{-1}_x (\bar{g}_{\Xi'})_x (\bar{g}^{\Xi'}_{\Xi})_x h_{\iota_\Xi(A_2)}(\gamma|_{[0,t]}) \\
&= h_{\iota_\Xi(A_1)}(\gamma|_{[0,t]})^{-1} (\bar{g}_\Xi)_x h_{\iota_\Xi(A_2)}(\gamma|_{[0,t]}) \\
&= \iota_\Xi(g_\Xi)(\gamma(t)),
\end{aligned}$$

   where we used again the transformation behaviour of ι_Ξ and $\iota_{\Xi'}$. Since γ was arbitrary in U and ι_Ξ is an embedding, we get $g_\Xi = g_{\Xi'}$ on $\pi^{-1}(U)$. Hence g is well defined. Moreover, since every g_Ξ is differentiable in the domain of differentiability of Ξ, we get the differentiability of g. The proof ends by $\iota_\Xi(g) = \bar{g}_\Xi$. □

Since we now know that the topological structure of the embeddings of $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{A}/\mathcal{G}$ as well as the structure of the action of $\mathcal{G}$ on $\mathcal{A}$ is completely independent of the choice of the trivialization, we will no longer worry about this and assume that we will have chosen in the next proofs, if necessary, silently some appropriate trivialization; however, the results will be independent. In particular, we will simply write $\mathcal{A} \subseteq \overline{\mathcal{A}}$ and A instead of $\iota_{\Xi}(A)$ etc.

4. Denseness

In this main section, we are going to study the properties of the embeddings of the smooth objects into their Ashtekar extensions. Before we come to the more difficult case of connections, we start with the rather easy investigation, when the space of regular gauge transformations is dense in that of generalized ones.

Theorem 4.1. *$\mathcal{G}$ is dense in $\overline{\mathcal{G}}$ iff* **G** *is connected.*

Proof.

- $\Rightarrow$ Let **G** be not connected and let g_1 and g_2 be contained in the different connected components $\mathbf{G}_{g_1}$ and $\mathbf{G}_{g_2}$, respectively. We define the open set $V := \mathbf{G}_{g_1} \times \mathbf{G}_{g_2}$, choose a differentiable connected trivialization $U \subseteq M$ and fix two points x_1 and x_2 in U. Then, for every edge γ connecting x_1 and x_2 in U, the set $\pi_\gamma^{-1}(V) = (\pi_{x_1} \times \pi_{x_2})^{-1}(V)$ is non-empty open in $\overline{\mathcal{G}}$ and contains no regular gauge transform in $\mathcal{G}$.
- $\Leftarrow$ Let **G** be connected. Then, obviously, for every hyph v and every $\vec{g} \in \mathbf{G}^{\#\mathbf{V}(v)}$ there is some regular gauge transform g with $g(x) = g_x$ for all $x \in \mathbf{V}(v)$. Hence $\pi_v(\mathcal{G}) = \mathbf{G}^{\#\mathbf{V}(v)} = \overline{\mathcal{G}}_v$ and Lemma A.1 yields the denseness. □

The remaining part of this section is devoted to the proof of the following theorem.

Theorem 4.2. *$\mathcal{A}$ is dense in $\overline{\mathcal{A}}$ precisely in the following cases*:

1. *in the analytic category for connected* **G**;
2. *in the immersive C^r category for connected and semisimple* **G**;
3. *in the non-immersive C^r category for trivial* **G**.

The same criterion is true for the denseness of $\mathcal{A}/\mathcal{G}$ in $\overline{\mathcal{A}}/\overline{\mathcal{G}}$.

We will prove this theorem case by case starting with the easiest one.

Lemma 4.3. *The denseness is given for trivial* **G**.

Proof. Obvious due to $\mathcal{A} = \overline{\mathcal{A}}$ and $\mathcal{A}/\mathcal{G} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$ for trivial **G**. □

The next lemma contains the remaining cases of denseness.

Lemma 4.4. *The denseness is given*

1. *in the analytic category for connected* **G**; *and*
2. *in the immersive C^r category for connected and semisimple* **G**.

Proof.

1. In the analytic case, the definition of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ using hyphs is equivalent to that using graphs [8]. Consequently, the denseness results for $\mathcal{A}/\mathcal{G}$ [17] and $\mathcal{A}$ [9] can be transferred immediately.
2. In the smooth immersive case, the definition of $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{A}/\mathcal{G}$ is equivalent to that using webs, where the denseness result for $\mathcal{A}$ has been proven in [13]. Since even $\pi_w(\mathcal{A}) = \mathbf{G}^{\# w}$ for all webs w [13], we get the denseness result by Corollary A.2 for $\mathcal{A}/\mathcal{G}$ as well.

   Originally, webs have been defined only for smooth, i.e. C^∞ paths. However, one sees quite immediately, that this definition and the corresponding subsequent theorems can be generalized to the case of arbitrary immersive C^r paths ($r > 0$). Therefore, the denseness results can be transferred as well. □

Now let us turn to the cases of non-denseness again starting with the two simplest ones.

Lemma 4.5. *The denseness is not given for non-connected* $\mathbf{G}$.

Note that using the standard identifications, we have $\mathcal{A}/\mathcal{G} = \mathcal{A}/\overline{\mathcal{G}}$.

Proof. Let α be a closed edge contained completely in a contractible neighbourhood of $m = \alpha(0) = \alpha(1)$ in M. Then (using some trivialization being smooth there) we have $\pi_\alpha(\mathcal{A}) \subseteq \mathbf{G}_0$, where $\mathbf{G}_0 \subset \mathbf{G}$ be the connected component of $e_{\mathbf{G}}$. On the other hand, since α is a hyph [8], we have $\overline{\mathcal{A}}_\alpha = \pi_\alpha(\overline{\mathcal{A}}) = \mathbf{G}$. The assertion now follows from Lemma A.1, because $\mathbf{G}_0$ is, of course, not dense in $\mathbf{G}$.

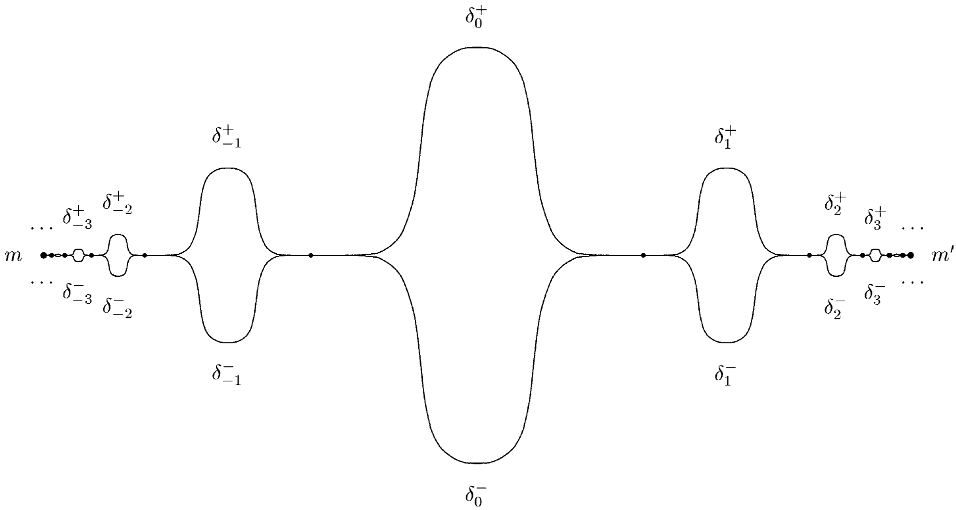
The proof for $\mathcal{A}/\mathcal{G}$ now follows from Corollary A.2: the action of $\overline{\mathcal{G}}_\alpha$ is just the adjoint action of $\mathbf{G}$ which leaves $\mathbf{G}_0$ invariant. □

Lemma 4.6. *The denseness is not given in the non-immersive smoothness category if* $\mathbf{G}$ *is non-trivial.*

This lemma has been shown for $\mathcal{A}$ already in [8]. We give here a slightly modified proof that includes both $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$.

Proof. Let γ be a closed, immersive path without self-intersections and $\gamma'(\tau) := \gamma(\tau^2)$. Then γ' is not equivalent to γ (cf. [11]) and not an immersion. Moreover, $\upsilon := \{\gamma, \gamma'\}$ is a hyph. However, since obviously $h_\gamma(A) = h_{\gamma'}(A)$ for all $A \in \mathcal{A}$, we have $\pi_\upsilon(\mathcal{A}) \subseteq \{(g, g) | g \in \mathbf{G}\}$ that is a non-dense subset of $\mathbf{G}^2 = \overline{\mathcal{A}}_\upsilon$ for non-trivial $\mathbf{G}$. Lemma A.1 yields the assumption for $\mathcal{A}$, Corollary A.2 that for $\mathcal{A}/\mathcal{G}$. In the last case, observe that $\overline{\mathcal{G}}_\upsilon$ is the adjoint action of $\mathbf{G}$ which leaves $\{(g, g) | g \in \mathbf{G}\}$ invariant. □

Now we come to the most difficult case. We will here reuse a certain example of paths (see Fig. 1) given in the paper [6] of Baez and Sawin. It has been used there to show that the direct transfer of the definition of spin networks from the analytic to the smooth category is not possible. Here we will exploit another property of these paths: they are independent as



$$\gamma_1 := \cdots\ \delta_{-4}^+\,\delta_{-3}^+\,\delta_{-2}^+\,\delta_{-1}^+\,\delta_0^+\,\delta_1^+\,\delta_2^+\,\delta_3^+\,\delta_4^+\ \cdots$$
$$\gamma_2 := \cdots\ \delta_{-4}^-\,\delta_{-3}^-\,\delta_{-2}^-\,\delta_{-1}^-\,\delta_0^-\,\delta_1^-\,\delta_2^-\,\delta_3^-\,\delta_4^-\ \cdots$$
$$\gamma_3 := \cdots\ \delta_{-4}^+\,\delta_{-3}^-\,\delta_{-2}^+\,\delta_{-1}^-\,\delta_0^+\,\delta_1^-\,\delta_2^+\,\delta_3^-\,\delta_4^+\ \cdots$$
$$\gamma_4 := \cdots\ \delta_{-4}^-\,\delta_{-3}^+\,\delta_{-2}^-\,\delta_{-1}^+\,\delta_0^-\,\delta_1^+\,\delta_2^-\,\delta_3^+\,\delta_4^-\ \cdots$$

Fig. 1. Paths used in the proof of Lemma 4.7.

graphs, but not holonomy-independent for abelian structure groups. This can be generalized to provide us with the following lemma.

Lemma 4.7. *The denseness is not given for non-semisimple connected* $\mathbf{G}$ *in the* C^r *smoothness category*.

Proof.

- Let us consider the map

$$\theta : \mathbf{G}^4 \to \mathbf{G}, \qquad \vec{g} \mapsto g_1 g_2 g_3^{-1} g_4^{-1}.$$

  Since $\mathbf{G}$ is supposed to be connected, it is isomorphic to $(\mathbf{G}_{\mathrm{ss}} \times U(1)^k)/\mathbf{N}$, where $\mathbf{G}_{\mathrm{ss}}$ is some semisimple Lie group, k is a natural number (by assumption $k \neq 0$) and $\mathbf{N}$ is a discrete central subgroup of $\mathbf{G}_{\mathrm{ss}} \times U(1)^k$. Now we define $K' := [\mathbf{G}_{\mathrm{ss}} \times \{e\}]_{\mathbf{N}} \subseteq \mathbf{G}$, where e is the identity in $U(1)^k$, and $K := \theta^{-1}(K') \subseteq \mathbf{G}^4$.
  - *K is Ad-invariant*. Let $\vec{g} \in K$, i.e. $\theta(\vec{g}) = [g_{\mathrm{ss}}, e]_{\mathbf{N}}$ for some $g_{\mathrm{ss}} \in \mathbf{G}_{\mathrm{ss}}$. Then for all $\tilde{g} = [\tilde{g}_{\mathrm{ss}}, \tilde{g}_{\mathrm{ab}}]_{\mathbf{N}} \in \mathbf{G}$, we have $\theta(\tilde{g}^{-1}\vec{g}\tilde{g}) = [\tilde{g}_{\mathrm{ss}}^{-1} g_{\mathrm{ss}} \tilde{g}_{\mathrm{ss}}, e]_{\mathbf{N}} \in K'$, hence $\tilde{g}^{-1}\vec{g}\tilde{g} \in \theta^{-1}(K') = K$.
  - *K is closed*. Obviously $\mathbf{G}_{\mathrm{ss}} \times \{e\} \subseteq \mathbf{G}_{\mathrm{ss}} \times U(1)^k$ is closed, hence $K' = [\mathbf{G}_{\mathrm{ss}} \times \{e\}]_{\mathbf{N}} \subseteq \mathbf{G}$ as well by the compactness of $\mathbf{N}$, i.e. K is closed by continuity of θ.

- ◦ $K \neq \mathbf{G}^4$. Let us assume $K = \mathbf{G}^4$. Then we have $\theta(\vec{g}) \in K'$ for all $\vec{g} \in \mathbf{G}^4$. This means, we always get

$$\theta([g_{1,\mathrm{ss}}, g_{1,\mathrm{ab}}]_{\mathbf{N}}, \ldots, [g_{4,\mathrm{ss}}, g_{4,\mathrm{ab}}]_{\mathbf{N}}) = [g_{1,\mathrm{ss}} g_{2,\mathrm{ss}} g_{3,\mathrm{ss}}^{-1} g_{4,\mathrm{ss}}^{-1}, g_{1,\mathrm{ab}} g_{2,\mathrm{ab}} g_{3,\mathrm{ab}}^{-1} g_{4,\mathrm{ab}}^{-1}]_{\mathbf{N}} = [g_{\mathrm{ss}}, e]_{\mathbf{N}}$$

  for some $g_{\mathrm{ss}} \in \mathbf{G}_{\mathrm{ss}}$. In particular, for all $\vec{g}_{\mathrm{ab}} \in (U(1)^k)^4$ there has to exist some $n = (n_{\mathrm{ss}}, n_{\mathrm{ab}}) \in \mathbf{N}$, such that $g_{1,\mathrm{ab}} g_{2,\mathrm{ab}} g_{3,\mathrm{ab}}^{-1} g_{4,\mathrm{ab}}^{-1} = n_{\mathrm{ab}}$. Then, by the finiteness of $\mathbf{N}$, also $U(1)^k$ has to be finite, which requires $k = 0$. But, then $\mathbf{G}$ is semisimple in contradiction to our assumption.
- ◦ *K is not dense in $\mathbf{G}^4$*. This follows simply from the fact that K is a closed proper subset of $\mathbf{G}^4$.

- Now we have a look at Fig. 1. Let the paths γ_i be given as indicated there, let γ be some path from m' to m, that does not intersect any of the paths γ_i and let $\alpha_i := \gamma_i \gamma$. Obviously, these four paths α_1 till α_4 form a hyph v with m being the free point for every α_i. However, although these paths are independent graph-theoretically, they are *not* independent w.r.t. regular connections: both $\gamma_1\gamma_2$ and $\gamma_4\gamma_3$ are paths, that run precisely once through each δ_j^+ and δ_j^- and precisely twice through γ. Consequently, the abelian parts of $h_A(\gamma_1\gamma_2)$ and $h_A(\gamma_4\gamma_3)$ coincide for every regular connection $A \in \mathcal{A}$ up to some n_{ab} in the abelian part of $\mathbf{N}$. Thus, $\theta(\pi_v(A)) = h_A(\gamma_1) h_A(\gamma_2) h_A(\gamma_3)^{-1} h_A(\gamma_4)^{-1} = [g_{\mathrm{ss}}, n_{\mathrm{ab}}]_{\mathbf{N}} = [g_{\mathrm{ss}} n_{\mathrm{ss}}^{-1}, e]_{\mathbf{N}} \in K'$ for some $g_{\mathrm{ss}} \in \mathbf{G}_{\mathrm{ss}}$ and some $n = (n_{\mathrm{ss}}, n_{\mathrm{ab}}) \in \mathbf{N}$, hence $\pi_v(\mathcal{A}) \subseteq \theta^{-1}(K') = K$. Since K is not dense in $\mathbf{G}^4 = \overline{\mathcal{A}}_v$, Lemma A.1 implies that $\mathcal{A}$ is not dense in $\overline{\mathcal{A}}$.
- The statement for $\mathcal{A}/\mathcal{G}$ follows now again by Corollary A.2, since $\overline{\mathcal{G}}_v$ is the adjoint action of $\mathbf{G}$ on $\mathbf{G}^4$ leaving K invariant. □

5. Discussion

The just proven lemma is in a certain sense a contrast to the usual expectation, that within the functional integral framework the classical theory should be a dense subset of the quantized theory (cf. as an easiest example the Wiener integral for the diffusion equation [16]). Let us, therefore, discuss how some modification of the definition of generalized connections could lead to the desired denseness of the regular connections.

1. *The number of paths is reduced*. This can most easily be done by sharpening the assumptions regarding the smoothness of paths. However, in view of applying the whole framework to quantum gravity, at least immersive C^∞-paths should be allowed. Then, of course, we are no longer able to couple theories to gravity which have non-semisimple structure groups.
2. *The number of independent paths is reduced*. One could try to plug the desired independence of paths into the equivalence relation of paths. This idea closely follows the idea of holonomy equivalence used in the first articles on Ashtekar connections [2,5]. Here two paths are said to be equivalent iff they have the same parallel transports for every regular connection. It is obvious that this way the proof of Lemma 4.7 can no longer be used: for instance, $\alpha_1\alpha_2$ and $\alpha_3\alpha_4$ would be equivalent in the abelian case. (But, note that

this is not true in the non-abelian case. Although here the holonomies are not arbitrarily selectable, they are not always equal.) However, the lack of denseness for non-abelian *and* non-semisimple structure groups remains true. This follows from the fact, that in the semisimple case $\pi_{\{\alpha_i\}_i} : \mathcal{A} \to \mathbf{G}^4$ is surjective, whence $\alpha_1\alpha_2$ and $\alpha_3\alpha_4$ cannot be equivalent at least if $\mathbf{G}$ contains a non-trivial semisimple part. In fact, it has been shown [13] that in the non-analytic immersed category for non-abelian structure groups two paths are holonomically equivalent iff they are graph-theoretically equivalent. Therefore, in these cases the usage of holonomy equivalence is not successful. Moreover, this method has the drawback that in the case of non-immersed paths it is probably very difficult to find at all concrete and explicit criteria for independence of paths.

3. *The range of the parallel transports is restricted*. This idea is precisely the basis for the definition of connections using webs. Namely, here—although done only for smooth and immersive paths—the space $\overline{\mathcal{A}}_{\mathrm{Web}}$ is not defined as a projective limit of all $\overline{\mathcal{A}}_w := \mathbf{G}^{\#w}$, but as a projective limit of all images $\mathcal{A}_w := \pi_w(\mathcal{A}) \subseteq \mathbf{G}^{\#w}$ of *regular* connections [7]. This way, automatically the surjectivity of $\pi_w : \mathcal{A} \to \overline{\mathcal{A}}_{\mathrm{Web}}$ is guaranteed, hence the denseness of $\mathcal{A}$ in $\overline{\mathcal{A}}_{\mathrm{Web}}$ as well. (The denseness follows, because first $\overline{\mathcal{A}}_{\mathrm{Web}} = \varprojlim_w \mathcal{A}_w$, second the set of all webs is directed and third the projections $\pi_w : \mathcal{A} \to \mathcal{A}_w$ are per definition surjective [9]. The denseness of $\mathcal{A}/\mathcal{G}$ in $\overline{\mathcal{A}}_{\mathrm{Web}}/\overline{\mathcal{G}}$ is now trivial.)

Unfortunately, none of the three possibilities discussed above is free of drawbacks such that a “final” decision about the definition of generalized connections can at most be given after studying more and concrete physical models.

6. Measure

In this final section, we generalize the theorem of Marolf and Mourão [14] about the Ashtekar–Lewandowski measure of $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$ to the case of arbitrary path categories considered here.

Theorem 6.1. *Both $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$ are contained in a set of Ashtekar–Lewandowski measure* 0 *provided* $\mathbf{G}$ *is non-trivial*.

In the case that $\mathbf{G}$ is trivial, we have $\mathcal{A} = \overline{\mathcal{A}}$ and $\mathcal{A}/\mathcal{G} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$ which means that the regular connection as well as the regular gauge orbit form a set of full measure 1.

Before we are going to prove the theorem, we note that Mourão et al. [15] were able to sharpen the statement above in the case of $\overline{\mathcal{A}}$ drastically: let e be some edge and e_s be the respective initial path of e w.r.t. the interval $[0, s]$ for $s \in [0, 1]$. Moreover, let

$$q_{\bar{A}} : [0, 1] \to \mathbf{G}, \qquad s \mapsto h_{\bar{A}}(e_s)$$

for all $\bar{A} \in \overline{\mathcal{A}}$. Then the set of all $\bar{A} \in \overline{\mathcal{A}}$ that possess just a single point in $[0, 1]$, where $q_{\bar{A}}$ is continuous, is contained in a μ_0-zero subset. This means, typically a generalized connection is nowhere continuous. Although the proof has been done in the analytic case, it can be transferred to the general case almost literally. However, that proof does not give a statement on the measure of $\mathcal{A}/\mathcal{G}$ such that we will not reuse it. Instead our proof is motivated by that of Marolf and Mourão: the only accessible quantities within the Ashtekar framework are parallel transports and holonomies. Therefore, one has to study how their behaviour is modified during the transition from regular to generalized connections. Typical for the regular case is—in total contrast to the generalized case—that parallel transports depend in a certain sense continuously on the paths. One can even prove that by means of a certain topology on $\mathcal{P}$ the regular connections can be identified with the continuous homomorphisms from $\mathcal{P}$ to M [12]. In particular, "small" paths have "small" parallel transports. A more detailed analysis [9] yields the well-known result (see e.g. [14]) that for every regular connection A the difference $h_A(\alpha) - e_{\mathbf{G}}$ is more or less proportional to the area enclosed by the sufficiently "round" loop α. This behaviour implies that the holonomies of a regular connection are trapped for shrinking α in a small area around $e_{\mathbf{G}}$ whose diameter decreases proportionally to $\|h_A(\alpha) - e_{\mathbf{G}}\|_\bullet$ and whose Haar measure consequently decreases as $|G_\alpha|^{\dim \mathbf{G}}$. However, generalized connections can even for very tiny α be anywhere in $\mathbf{G}$.

Altogether we have the following proof.

Proof.

- Let first $\dim \mathbf{G} = 0$. Then (in an appropriate neighbourhood of m) $h_\alpha(A) = e_{\mathbf{G}}$ for all regular A. Let now $(\alpha_i)_{i\in\mathbb{N}} \subseteq \mathcal{HG}$ be a sequence of mutually non-intersecting closed edges having base point m. Then

  $$\mu_0(\mathcal{A}) \leq \mu(\pi^{-1}_{\{\alpha_1,\ldots,\alpha_i\}}(\{e_{\mathbf{G}}\})) = \mu_{\mathrm{Haar}}(\{e_{\mathbf{G}}\})^i = (\#\mathbf{G})^{-i}$$

  for all $i \in \mathbb{N}$. Since $\mathbf{G}$ is non-trivial, hence $\#\mathbf{G} \geq 2$, we have $\mu_0(\mathcal{A}) = 0$. Analogously, we get $\mu(\mathcal{A}/\mathcal{G}) = 0$.
- Let now $\dim \mathbf{G} > 0$. We consider $\mathbf{G}$ as a subset of some $U(n) \subseteq \mathrm{Gl}_{\mathbb{C}}(n) \subseteq \mathbb{C}^{n\times n}$ (and so $\mathfrak{g} \subseteq \mathfrak{gl}_{\mathbb{C}}(n) = \mathbb{C}^{n\times n}$ as well), choose some $\mathrm{Ad}\,\mathbf{G}$-invariant norm $\|\cdot\|_\bullet$ on $\mathbb{C}^{n\times n}$ and define $B_\varepsilon(e_{\mathbf{G}}) := \{g \in \mathbf{G} \mid \|g - e_{\mathbf{G}}\|_\bullet < \varepsilon\}$ for all $\varepsilon \in \mathbb{R}_+$. (For instance, $\|D\|_\bullet := \sup_{\vec{x}\in\mathbb{C}^n, \|\vec{x}\|=1} \|D\vec{x}\|$, $D \in \mathbb{C}^{n\times n}$, is $\mathrm{Ad}\,\mathbf{G}$-invariant due to the unitarity of any compact group.) Obviously, $B_\varepsilon(e_{\mathbf{G}})$ is always an $\mathrm{Ad}\,\mathbf{G}$-invariant set.

  Next, we choose some chart mapping $\kappa : M \supseteq U \to \kappa(U)$, such that $m \in U$. W.l.o.g. $\kappa(U) \subseteq \mathbb{R}^{\dim M}$ be bounded and $\kappa(m) = 0$. We assign to U the Euclidean metric and choose some surface $H \subseteq U$ spanned by two coordinates.

  Finally, we assume that the chart image of every $\alpha \in \mathcal{HG}$ used below is a circle in $\kappa(H) \subseteq \mathbb{R}^2$. The area of the domain enclosed by α in H be $|G_\alpha|$.
- Now we define for all α and all real $r \in \mathbb{R}_+$, the set

  $$U_{\alpha,r} := \pi_\alpha^{-1}(B_{r|G_\alpha|}(e_{\mathbf{G}})) \subseteq \overline{\mathcal{A}}$$

  being $\overline{\mathcal{G}}$-invariant by the Ad-invariance of $B_\varepsilon(e_{\mathbf{G}})$. By the appendix in [10], we have $\mu_0(U_{\alpha,r}) = \mu_{\mathrm{Haar}}(B_{r|G_\alpha|}(e_{\mathbf{G}})) \leq \mathrm{const}(r|G_\alpha|)^{\dim \mathbf{G}}$. Hence, $\mu_0(U_{\alpha,r})$ goes to 0, in particular, for $|G_\alpha| \downarrow 0$.
- Let now $(\alpha_i)_{i\in\mathbb{N}}$ be some sequence of circles with $|G_{\alpha_i}| \downarrow 0$, such that each two of them have precisely m as common point. We define

  $$U_r := \bigcap_{i\in\mathbb{N}} U_{\alpha_i,r}.$$

  Obviously $\mu_0(U_r) \leq \inf_i \{\mu_0(U_{\alpha_i,r})\} = 0$.

- On the other hand, for every $A \in \mathcal{A}$ there is some $c_A \in \mathbb{R}_+$ with $A \in U_{\alpha,c_A} \equiv \pi_\alpha^{-1}(B_{c_A|G_\alpha|}(e_{\mathbf{G}}))$ for all circles α (cf. Appendix in [10]). Hence, $A \in U_{c_A}$. Therefore, $U := \bigcup_{r\in\mathbb{N}_+} U_r$ is obviously a μ_0-zero subset containing $\mathcal{A}$. Since U is even $\overline{\mathcal{G}}$-invariant, $U/\overline{\mathcal{G}}$ is a μ_0-zero subset as well, now containing $\mathcal{A}/\overline{\mathcal{G}} = \mathcal{A}/\mathcal{G}$. □

We note finally, that the just proven support property is typical for the description of physical theories in terms of functional integrals. For instance, it is well known that in the Wiener-integral case the classical configuration space of all C^1-paths is a zero subset in its completion (cf. [16]). But, this is to be expected, since otherwise the measure on the completion would define a non-trivial, physically "reasonable" measure on the classical configuration space as well, although the existence of such a measure usually is seen to be unlikely.

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Appendix A. Denseness criteria

Let A be a set and $\leq$ be a partial ordering on A. Next, let X_a be a topological space for each $a \in A$ and $\pi^{a_2}_{a_1} : X_{a_2} \to X_{a_1}$ for all $a_1 \leq a_2$ be a continuous and surjective map with $\pi^{a_2}_{a_1} \circ \pi^{a_3}_{a_2} = \pi^{a_3}_{a_1}$ if $a_1 \leq a_2 \leq a_3$. The corresponding projective limit $\lim_{\leftarrow a\in A} X_a$ is denoted by $\overline{\mathcal{X}}$. Furthermore, let $\pi_a : \overline{\mathcal{X}} \to X_a$ be the usual continuous projection on the a-component and X be some subset of $\overline{\mathcal{X}}$. Finally, let A be directed, i.e. for any two $a', a'' \in A$, there is an $a \in A$ with $a', a'' \leq a$.

Lemma A.1. *X is dense in $\overline{\mathcal{X}}$ iff $\pi_a(X)$ is dense in X_a for all $a \in A$.*

The $\Leftarrow$-direction has already been proven in [8]. We quote it for completeness.

Proof.

$\Rightarrow$ Let $a \in A$ be arbitrary and let $U_a \subseteq X_a$ be open. Then $\pi_a^{-1}(U_a)$ is open in $\overline{\mathcal{X}}$. Hence, there is an $x \in X$ with $x \in \pi_a^{-1}(U_a)$. Consequently, $\pi_a(x) \in \pi_a(\pi_a^{-1}(U_a)) \subseteq U_a$.

$\Leftarrow$ Let $U \subseteq \overline{\mathcal{X}}$ be open and non-empty, i.e. $U \supseteq \bigcap_i \pi_{a_i}^{-1}(V_i) \neq \emptyset$ with open $V_i \subseteq X_{a_i}$ and finitely many $a_i \in A$. Since A is directed, there is an $a \in A$ with $a_i \leq a$ for all i and thus $U \supseteq \pi_a^{-1}(\bigcap_i (\pi^a_{a_i})^{-1}(V_i))$ with non-empty $V := \bigcap_i (\pi^a_{a_i})^{-1}(V_i) \subseteq X_a$. V is open because $\pi^a_{a_i}$ is continuous. Since $\pi_a(X)$ is dense in X_a, there is an $x \in X$ with $\pi_a(x) \in V$ and so $\pi_{a_i}(x) \in V_i$ for all i, hence $x \in U$. □

Now, let additionally $\overline{\mathcal{G}} := \lim_{\substack{\leftarrow \\ a\in A}} G_a$ be some projective limit of compact topological groups G_a acting continuously and compatibly on the corresponding compact Hausdorff spaces X_a. Moreover, for both projective systems all the projections π_a be surjective. For a precise definition of projective-limit group actions, see [4,3,9].

Corollary A.2. *$X/\overline{\mathcal{G}}$ is dense in $\overline{\mathcal{X}}/\overline{\mathcal{G}}$ iff $\pi_a(X)/G_a$ is dense in X_a/G_a for all $a \in A$. In particular, we have*

- *denseness if $\pi_a(X) = X_a$ for all $a \in A$; and*
- *non-denseness if $\pi_a(X)$ is for all $a \in A$ contained in some G_a-invariant and non-dense subset of X_a.*

Proof. We define $\overline{\mathcal{X}/\mathcal{G}} := \varprojlim_{a\in A} X_a/G_a$. Then, by the assumptions, the map

$$\boldsymbol{\phi} : \overline{\mathcal{X}}/\overline{\mathcal{G}} \to \overline{\mathcal{X}/\mathcal{G}}, \qquad [(x_a)_{a\in A}] \mapsto ([x_a])_{a\in A}$$

is a well-defined homeomorphism [9]. By the lemma above, $X/\overline{\mathcal{G}}$ is now dense in $\overline{\mathcal{X}}/\overline{\mathcal{G}}$ iff $\pi_a(\boldsymbol{\phi}(X/\overline{\mathcal{G}})) = \pi_a(X)/G_a$ is dense in X_a/G_a for all $a \in A$.

To prove the non-denseness in the special case that $\pi_a(X)$ is for all $a \in A$ contained in some G_a-invariant and non-dense subset of X_a, use the fact that the canonical projections $X_a \to X_a/G_a$ are always open. The other special case is trivial. □

The above assumptions for the projective limits are fulfilled for $\overline{\mathcal{X}} = \overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$ defined using hyphs as label set A [11,8,9].

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